Nonstationary stochastic analysis of flow in a heterogeneous semiconfined aquifer

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Abstract. In this study, we investigate two-dimensional flow through a heterogeneous, semiconfined aquifer. In the presence of leakage, the mean flow varies in space and the fluctuations of the flow become nonstationary spatially. Such a situation calls for a nonstationary stochastic approach since the classical stationary stochastic approaches are no longer appropriate. We make use of a nonstationary spectral method to account for such nonstationarities in finite semiconfined aquifers. Analytical expressions are obtained for head and specific discharge covariances that account for the spatial variability in the mean flow but neglect the contributions of the finite boundaries. Closed-form analytical expressions for the variances of hydraulic head and specific discharge are derived. The statistical structures of the head and specific discharge fields are investigated in terms of the leakage factor and the spatial structure of hydraulic conductivity field. Results based on the nonstationary approach show that the stationary assumption is inappropriate even for a small leakage.

1. Introduction

Formation spatial variabilities have been the focus of many efforts to understand flow and transport in heterogeneous systems. Hydraulic head and velocity (specific discharge) are strongly influenced by inherent formation properties, e.g., hydraulic conductivity and porosity. When the hydraulic conductivity field is regarded as a random space function, the various flow quantities like hydraulic head, specific discharge and velocity also become random space functions. For confined aquifers, many studies have focused on relating the statistical moments of head and velocity to those of the aquifer properties [e.g., *Dagan*, 1982, 1984; *Gelhar and Axness*, 1983; *Neuman et al.*, 1987]. In a shallow aquifer system, the confining beds of a confined aquifer are never truly confined. When leakage through the confining beds cannot be neglected, it is referred to as a semiconfined aquifer. In such a semiconfined aquifer, the covariances of head and specific discharge depend on the statistical structure of hydraulic conductivity, the leakage factor as well as its hydraulic boundary conditions. Due to the complex nature of flow in semiconfined aquifers, the flow quantities are usually location dependent and thus spatially nonstationary. Recently, *Zhu* [1998] and *Zhu and Sykes* [2000] derived analytical solutions for the head and specific discharge covariances by assuming stationarity for the flow of the semiconfined aquifer. In this study, we make use of a nonstationary spectral method [*Li and McLaughlin*, 1991, 1995] to account for statistical nonstationarities and their effects on flow in a shallow semiconfined aquifer. In particular, the objectives of this study are to (1) present the results of the nonstationary approach to quantify the uncertainty of the head and specific discharge in the semiconfined aquifer and (2) compare the results of this nonstationary study with those from a stationary approach.

This manuscript is organized into six sections. The conceptual model is discussed in section 2, and the stationary and nonstationary approaches are discussed in section 3. The head and specific discharge covariances are presented in section 4, and the results and the conclusions are discussed in the last two sections.

2. Conceptual Model

An aquifer is called semiconfined if its confining beds are not truly impermeable and flow (leakage) occurs through these beds (Figure 1). Flow is essentially horizontal in a shallow semiconfined aquifer if the semiconfined aquifer is horizontal, the lower layer is impermeable, and the head is constant above the upper semipermeable layer. Flow in such a situation satisfies the following governing equation [*Bear*, 1972],

$$\frac{\partial}{\partial x_i}\left[K(\mathbf{x})\frac{\partial h(\mathbf{x})}{\partial x_i}\right]=K(\mathbf{x})\alpha^2[h(\mathbf{x})-h^*] \tag{1}$$

where summation for repeated indices "i" is implied with i=1, 2 (here and throughout the text subscript i stands for a component of a vector), $\mathbf{x} = (x_1, x_2)^T$ is the vector of coordinate (where T denoting transpose), $h(\mathbf{x})$ is the head in the aquifer, $K(\mathbf{x})$ is the hydraulic conductivity, which is treated as a random space function with known statistical moments, h^* is the head in the aquifer above the semipermeable layer, and $\alpha=\sqrt{K^*/(KHH^*)}$, is the so-called leakage factor [*Zhu*, 1998]. K^* and H^* are the hydraulic conductivity and thickness of the semipermeable layer, and H is the thickness of the main aquifer. In this study, K^* is assumed much smaller, at least by a factor of *10*, than K [*Strack*, 1989], then α is considered to be small as a deterministic constant. h^* is also assumed to be a deterministic constant [*Zhu*, 1998]. Equation (1) can be rewritten in terms of the relative head $h_r(\mathbf{x}) = h(\mathbf{x}) - h^*$,

$$
\begin{aligned}
&\frac{\partial}{\partial x_i}\left[K(\mathbf{x})\frac{\partial h_r(\mathbf{x})}{\partial x_i}\right] - K(\mathbf{x})\alpha^2 h_r(\mathbf{x}) = 0 \\
&h_r(\mathbf{x}) = H_0, \qquad x_1 = 0 \\
&h_r(\mathbf{x}) = H_L, \qquad x_1 = L \\
&q_2(\mathbf{x}) = K(\mathbf{x})\frac{\partial h_r(\mathbf{x})}{\partial x_2} = 0, \qquad \mathbf{x} \in \Gamma_N
\end{aligned}
\tag{2}
$$

where Γ_N denotes the no-flow boundary segments, and q_2 is the specific discharge in x_2 direction.

3. Spectral Methods

In this section, we show how to derive moment equations for hydraulic head and specific discharge using stationary and nonstationary spectral methods. It will be shown that the general formulation derived with the nonstationary approach reduces to the stationary representation under the assumption of constant mean head gradient.

As commonly done in the literature, we work with the log transformed hydraulic conductivity ln$K$, which may be decomposed into its mean and fluctuation as

$$f(\mathbf{x}) = \ln K(\mathbf{x}) = \langle f(\mathbf{x}) \rangle + f'(\mathbf{x}) \tag{3}$$

where and in the following text angular brackets < > indicate a mathematical expectation (ensemble mean), and the primed quantity is the zero mean fluctuation. Expanding the relative head h_r and the specific discharge q into the following formal series:

$$h_r(\mathbf{x}) = h_r^{(0)}(\mathbf{x}) + h_r^{(1)}(\mathbf{x}) + h_r^{(2)}(\mathbf{x}) + ... \tag{4}$$

$$q(\mathbf{x}) = q^{(0)}(\mathbf{x}) + q^{(1)}(\mathbf{x}) + q^{(2)}(\mathbf{x}) + ... \tag{5}$$

where $h_r^{(n)}(\mathbf{x})$ and $q_i^{(n)}(\mathbf{x})$ (with n = 0, 1, 2, ⋯) are, in a statistical sense, terms of n-th order in σ_f, which is the standard deviation of f.

Substitution of (3)-(5) into (2) and collecting terms of the same order leads to [*Zhang, 2002*],

$$\begin{aligned}
&\frac{\partial^2 h_r^{(0)}(\mathbf{x})}{\partial x_i^2} - \alpha^2 h_r^{(0)}(\mathbf{x}) = 0 \\
&h_r^{(0)}(\mathbf{x}) = H_0, \qquad x_1 = 0 \\
&h_r^{(0)}(\mathbf{x}) = H_L, \qquad x_1 = L \\
&n_2(\mathbf{x}) \frac{\partial h_r^{(0)}(\mathbf{x})}{\partial x_2} = 0, \qquad \mathbf{x} \in \Gamma_N
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
&\frac{\partial^2 h_r^{(1)}(\mathbf{x})}{\partial x_i^2} - \alpha^2 h_r^{(1)}(\mathbf{x}) = J_{xi}(x_i) \frac{\partial f'(\mathbf{x})}{\partial x_i} \\
&h_r^{(1)}(\mathbf{x}) = 0, \qquad x_1 = 0 \\
&h_r^{(1)}(\mathbf{x}) = 0, \qquad x_1 = L \\
&n_2(\mathbf{x}) \frac{\partial h_r^{(1)}(\mathbf{x})}{\partial x_2} = 0, \qquad \mathbf{x} \in \Gamma_N
\end{aligned} \tag{7}$$

where $J_{xi}(x_i) = -\partial h_r^{(0)}(\mathbf{x}) / \partial x_i$ is the (negative) mean head gradient, and $n_2(\mathbf{x})$ is an outward unit vector normal to the boundary. It can be shown [Zhang, 2002] that $< h_r^{(0)}(\mathbf{x}) > = h_r^{(0)}(\mathbf{x})$ and $< h_r^{(1)}(\mathbf{x}) > = 0$. Hence, to first order in σ_f the fluctuation of the relative head is equal to $h_r^{(1)}(\mathbf{x})$. For simplicity, we use $h'(\mathbf{x})$ to denote the first order head fluctuation $h_r^{(1)}(\mathbf{x})$. Likewise, we denote the first order specific discharge fluctuation as $q_i'(\mathbf{x})$.

If the dependent variables (e.g., the head fluctuation $h'(\mathbf{x})$ and the specific discharge $q_i'(\mathbf{x})$) are spatially stationary, which is equivalent to uniform mean flow in unbounded domains under the condition that the independent variable ($f(\mathbf{x}) = \ln K(\mathbf{x})$) is stationary, the fluctuations f', h' and q_i' can be expressed by the following stochastic Fourier-Stieltjes integral representations [*Lumley and Panofsky*, 1964; *Bakr et al.*, 1978],

$$f'(\mathbf{x}) = \int \exp(\iota\mathbf{k}\cdot\mathbf{x})dZ_f(\mathbf{k}) \tag{8}$$

$$h'(\mathbf{x}) = \int \exp(\iota\mathbf{k}\cdot\mathbf{x})dZ_h(\mathbf{k}) \tag{9}$$

$$q_i'(\mathbf{x}) = \int \exp(\iota\mathbf{k}\cdot\mathbf{x})dZ_{q_i}(\mathbf{k}) \tag{10}$$

where $\mathbf{k} = (k_1, \ldots, k_d)^T$ is the wave number space vector (where d being the number of space dimensions), $\iota \equiv \sqrt{-1}$, and $dZ_f(\mathbf{k})$, $dZ_h(\mathbf{k})$, and $dZ_{q_i}(\mathbf{k})$ are the complex Fourier increments of the fluctuations at $\mathbf{k}$. The integration is d-fold from $-\infty$ to ∞. The stochastic Fourier-Stieltjes integral has the following properties, using dZ_f as an example,

$$< dZ_f(\mathbf{k}) >= 0 \tag{11}$$

$$< dZ_f(\mathbf{k})dZ_f^*(\mathbf{k}') >= S_{ff}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')d\mathbf{k}d\mathbf{k}'$$

where and in the following text the superscript * indicates the corresponding conjugate, thus dZ_f^* is the complex conjugate of dZ_f, $S_{ff}(\mathbf{k})$ is the spectrum (i.e., spectral density function) of f if it is integrable, and δ is Dirac delta function. The first equation of (11) indicates zero mean for the random increment dZ_f and the second one states the so-called orthogonality property of dZ_f.

Here let us consider the solution of the (zeroth-order) mean head in Equation (6) in a two-dimensional, horizontal bounded domain of two parallel constant head and two parallel no-flow boundaries (Figure 1). It can be verified [*Zhang, 2002*] that

$$< h_r^{(0)}(\mathbf{x}) >= \frac{H_L - H_0 \exp(-\alpha L)}{\exp(\alpha L) - \exp(-\alpha L)} \exp(\alpha x_1) + \frac{H_0 \exp(\alpha L) - H_L}{\exp(\alpha L) - \exp(-\alpha L)} \exp(-\alpha x_1) \quad (12)$$

Thus the mean head gradient is given as

$$J_{x_1}(x_1) = A \exp(\alpha x_1) + B \exp(-\alpha x_1) \quad (13)$$

$$J_{x_2}(x_2) = 0 \quad (14)$$

where $A = -\alpha[H_L - H_0 \exp(-\alpha L)]/[\exp(\alpha L) - \exp(-\alpha L)]$, and

$B = \alpha[H_0 \exp(\alpha L) - H_L]/[\exp(\alpha L) - \exp(-\alpha L)]$.

Figure 2 illustrates the mean relative head and its mean (negative) spatial gradient as functions of the leakage factor. The domain is of *10 m* x *10 m* with the following boundary conditions: $h_r = -1.0\ m$ at $x_1=0$, and $h_r = -1.1\ m$ at $x_1=10\ m$. The curves are plotted in the x_1 direction (longitudinally) since there is no variation of the mean flow in the x_2 direction (transversely). The results are shown for leakage factor $\alpha = 0, 0.01, 0.02$, and $0.04\ m^{-1}$. The choices of these leakage factors are consistent with the assumption required in Eq. (1). As pointed out by Strack [1989], the hydraulic conductivity K^* of the leaky layer must be much less (at least by a factor of 10) than the hydraulic conductivity K of the aquifer to ensure the validity of Eq. (1). The possible scenario of $K^*/K = 0.1$, $H^* = 1\ m$ and $H = 10\ m$ gives $\alpha = 0.1\ m^{-1}$, which is larger than the above values. As a matter of fact, given $K^*/K = 0.1$ any combination of H^* and H under the condition $H^*H \leq 62.5$

m^2 leads to the leakage factor $\alpha \geq 0.04\ m^{-1}$. It is seen from Figure 2 that the mean relative head is generally not a linear function of x_1 and the mean head gradient is thus not constant in space (along the x_1 direction). Only at the small-α limit, i.e., $\alpha \rightarrow 0$, is the mean head gradient constant, which corresponds to the situation of confined flow between two impermeable strata (layers). When $J_{x1}(x_1)$ is a function of space, the head fluctuation will also be space dependent and thus nonstationary even in an unbounded domain.

3.1 Nonstationary Spectral Method

In general, for the problem of flow in a semiconfined aquifer a nonstationary approach is appropriate. In a recent work, *Zhang* [2002] discussed a number of nonstationary stochastic methods, most of which are based on real space representations. In this study, we make use of a nonstationary spectral representation, in which the nonstationary head fluctuation is expressed through a generalized spectral representation [*Li and McLaughlin*, 1991, 1995],

$$h'(\mathbf{x}) = \int \phi_{hf}(\mathbf{x},\mathbf{k}) dZ_f(\mathbf{k}) \tag{15}$$

where $\phi_{hf}(\mathbf{x},\mathbf{k})$ is a transfer function to be given. Then the covariance between head at location $\mathbf{x}$ and $\mathbf{x}'$ is given by

$$\begin{aligned} C_{hh}(\mathbf{x}, \mathbf{x}') &= < h'(\mathbf{x})h'(\mathbf{x}') > = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{hf}(\mathbf{x},\mathbf{k})\phi^*_{hf}(\mathbf{x}',\mathbf{k}') < dZ_f(\mathbf{k})dZ^*_f(\mathbf{k}') > \\ &= \int \phi_{hf}(\mathbf{x},\mathbf{k})\phi^*_{hf}(\mathbf{x}',\mathbf{k})S_{ff}(\mathbf{k})d\mathbf{k} \end{aligned} \tag{16}$$

where $S_{ff}(\mathbf{k})$ is spectral density function of the hydraulic conductivity field. A specific form of this spectral density function will be given in the next section. The statistical moments of specific discharge can be related to those of the hydraulic conductivity and hydraulic head using Darcy's law

$$q_i(\mathbf{x}) = -K(\mathbf{x})\frac{\partial h(\mathbf{x})}{\partial x_i} = -\exp[< f(\mathbf{x}) > + f'(\mathbf{x})]\left[\frac{\partial h^{(0)}(\mathbf{x})}{\partial x_i} + \frac{\partial h^{(1)}(\mathbf{x})}{\partial x_i} + \ldots\right] \tag{17}$$

Expanding the exponential function in (17) and collecting terms of the same order, one obtains the specific discharge fluctuation, to first-order, as

$$q_i'(\mathbf{x}) = -K_0\left(f^{(1)}(\mathbf{x})\frac{\partial h^{(0)}(\mathbf{x})}{\partial x_i} + \frac{\partial h^{(1)}(\mathbf{x})}{\partial x_i}\right) \tag{18}$$

where $K_0 = \exp(<f(\mathbf{x})>)$ is the geometric mean hydraulic conductivity. With this, the covariance functions of the specific discharge are given in Section 4.

With (13)-(14), the first-order head fluctuation equation (7) can be written as

$$\frac{\partial^2 h_r^{(1)}(\mathbf{x})}{\partial x_i^2} - \alpha^2 h_r^{(1)}(\mathbf{x}) = \left[A\exp(\alpha x_1) + B\exp(-\alpha x_1)\right]\frac{\partial f'(\mathbf{x})}{\partial x_i} \tag{19}$$

The transfer function $\phi_{hf}(\mathbf{x},\mathbf{k})$ can be derived by substituting (8) and (15) into (19),

$$\int\{\frac{\partial^2\phi_{hf}(\mathbf{x},\mathbf{k})}{\partial x_i^2} - \alpha^2\phi_{hf}(\mathbf{x},\mathbf{k}) - [A\exp(\alpha x_1) + B\exp(-\alpha x_1)]\frac{\partial \exp(\iota\mathbf{k}\cdot\mathbf{x})}{\partial x_i}\}dZ_f(\mathbf{k}) = 0 \tag{20}$$

Multiplying Equation (20) with its complex conjugate and taking expectation [*Li and McLaughlin*, 1995] leads to

$$\int\left|\frac{\partial^2\phi_{hf}(\mathbf{x},\mathbf{k})}{\partial x_i^2} - \alpha^2\phi_{hf}(\mathbf{x},\mathbf{k}) - [A\exp(\alpha x_1) + B\exp(-\alpha x_1)]\frac{\partial \exp(\iota\mathbf{k}\cdot\mathbf{x})}{\partial x_i}\right|^2 S_{ff}(\mathbf{k})d\mathbf{k} = 0 \tag{21}$$

where vertical bars indicate absolute value. The necessary and sufficient condition for equation (21) to hold for any arbitrary $S_{ff}(\mathbf{k})$ is

$$\frac{\partial^2 \phi_{hf}(\mathbf{x},\mathbf{k})}{\partial x_i^2} - \alpha^2 \phi_{hf}(\mathbf{x},\mathbf{k}) = [A\exp(\alpha x_1) + B\exp(-\alpha x_1)]\frac{\partial \exp(\iota \mathbf{k}\cdot\mathbf{x})}{\partial x_1} \tag{22}$$

Equation (22) is the so-called modified Helmholtz equation. Because of the special exponential forms of the right-hand side of (22), we solve it by inspection and express the solution as

$$\phi_{hf}(\mathbf{x},\mathbf{k}) = C(\mathbf{k})\exp(\alpha x_1 + \iota \mathbf{k}\cdot\mathbf{x}) + D(\mathbf{k})\exp(-\alpha x_1 + \iota \mathbf{k}\cdot\mathbf{x}) \tag{23}$$

where $C(\mathbf{k})$ and $D(\mathbf{k})$ are to be determined. Substituting (23) into (22) and solving for the unknown coefficients (by comparing terms with $\exp(\alpha x_1)$ and $\exp(-\alpha x_1)$ on both sides), the resulting transfer function has the following form,

$$\begin{aligned}
\phi_{hf}(\mathbf{x},\mathbf{k}) &= -\frac{\iota k_1 A}{k^2 - 2\iota\alpha k_1}\exp(\alpha x_1 + \iota k_i x_i) - \frac{\iota k_1 B}{k^2 + 2\iota\alpha k_1}\exp(-\alpha x_1 + \iota k_i x_i) \\
&= \frac{-k_1}{k^4 + 4\alpha^2 k_1^2}[A(-2\alpha k_1 + \iota k^2)\exp(\alpha x_1) + B(2\alpha k_1 + \iota k^2)\exp(-\alpha x_1)] \times \exp(\iota k_i x_i) \\
&= \frac{-k_1}{k^4 + 4\alpha^2 k_1^2}\{2\alpha k_1[-A\exp(\alpha x_1) + B\exp(-\alpha x_1)] \\
&\quad + \iota k^2[A\exp(\alpha x_1) + B\exp(-\alpha x_1)]\}\exp(\iota k_i x_i) \\
&= \frac{-k_1}{k^4 + 4\alpha^2 k_1^2}[2\alpha^2 k_1 h_r^0(x_1) + \iota k^2 J_{x1}(x_1)]\exp(\iota k_i x_i)
\end{aligned} \tag{24}$$

where $h_r^0(\mathbf{x})$ is a short notation for $< h_r^{(0)}(\mathbf{x}) >$. In deriving (24), we have made the assumption that the effect of finite boundaries on the head fluctuation is negligible. This simplifying assumption is made for the purpose of obtaining analytical solutions. Hence, the head fluctuation can be expressed as

$$\begin{aligned}
h^{(1)}(\mathbf{x}) &= \int \phi_{hf}(\mathbf{x},\mathbf{k})dZ_f(\mathbf{k}) \\
&= \int \frac{-k_1}{k^4 + 4\alpha^2 k_1^2}[2\alpha^2 k_1 h_r^0(x_1) + \iota k^2 J_{x1}(x_1)]\exp(\iota k_i x_i)dZ_f(\mathbf{k})
\end{aligned} \tag{25}$$

With (25), the spectral expression for the covariance between head at location **x** and **x'** is given by

$$C_{hh}(\mathbf{x}, \mathbf{x}') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{k_1^2}{(k^4 + 4\alpha^2 k_1^2)^2} \{4\alpha^4 k_1^2 h_r^0(x_1) h_r^0(x_1') + k^4 J_{x1}(x_1) J_{x1}(x_1')\} S_{ff}(\mathbf{k}) \cos(k_1\xi_1 + k_2\xi_2) dk_1 dk_2 \tag{26}$$

where $\boldsymbol{\xi} = \mathbf{x}' - \mathbf{x}$ is a displacement vector with components ξ_1 and ξ_2.

The nonstationary specific discharge fluctuation can be obtained by substituting (8) and (13), (14) and (25) into the equation (18),

$$\begin{aligned} q_i'(\mathbf{x}) &= -K_0 \{\int \exp(\iota k_i x_i) dZ_f(\mathbf{k}) [-J_{xi}(x_1)] \\ &+ \frac{\partial}{\partial x_i} [\int \frac{-k_1}{k^4 + 4\alpha^2 k_1^2} [2\alpha^2 k_1 h_r^0(x_1) + \iota k^2 J_{x1}(x_1)] \exp(\iota k_i x_i) dZ_f(\mathbf{k})]\} \\ &= K_0 \frac{k_1}{k^4 + 4\alpha^2 k_1^2} \{\int \exp(\iota k_i x_i) dZ_f(\mathbf{k}) \frac{k^4 + 4\alpha^2 k_1^2}{k_1} \delta_{i1} J_{x1}(x_1) \\ &+ [\int -k^2 k_i J_{x1}(x_1) - 2\alpha^2 k_1 J_{x1}(x_1) \\ &+ \iota 2\alpha^2 k_1 k_i h_r^0(x_1) - \iota\alpha^2 k^2 h_r^0(x_1)] \exp(\iota k_i x_i) dZ_f(\mathbf{k})\} \\ &= K_0 \frac{k_1}{k^4 + 4\alpha^2 k_1^2} \{\int J_{x1}(x_1) [\delta_{i1} \frac{k^4 + 4\alpha^2 k_1^2}{k_1} - k_i k^2 - 2\alpha^2 k_1] \\ &+ \iota\alpha^2 h_r^0(x_1) [2k_1 k_i - k^2]\} \exp(\iota k_i x_i) dZ_f(\mathbf{k})\} \end{aligned} \tag{27}$$

The covariance functions of the specific discharge can then be obtained as

$$\begin{aligned} C_{q_i q_j}(\mathbf{x},\mathbf{x}') &= \langle q'_i(\mathbf{x}) q'_j(\mathbf{x}') \rangle \\ &= \int K_0^2 \frac{k_1^2}{(k^4 + 4\alpha^2 k_1^2)^2} \{\alpha^4 h_r^0(x_1) h_r^0(x_1') [2k_1 k_i - k^2][2k_1 k_j - k^2] \\ &+ J_{x1}(x_1) J_{x1}(x_1') [\delta_{i1} \frac{k^4 + 4\alpha^2 k_1^2}{k_1} - k_i k^2 - 2\alpha^2 k_1] \\ &[\delta_{j1} \frac{k^4 + 4\alpha^2 k_1^2}{k_1} - k_j k^2 - 2\alpha^2 k_1]\} S_{ff}(\mathbf{k}) \cos(k_1\xi_1 + k_2\xi_2) d\mathbf{k} \end{aligned} \tag{28}$$

3.2. Stationary Limit

The formulations derived in Section 3.1 are for generally nonstationary flows given that the log hydraulic conductivity field is stationary. As shown in Figure 2, for the limiting case where $\alpha \to 0$ the mean gradient is approximately constant, i.e., $J_{x1}(\mathbf{x}) \approx J_0$. Under this condition, the equation governing the transfer function $\phi_{hf}(\mathbf{x},\mathbf{k})$ can be rewritten from (22) as

$$\frac{\partial^2 \phi_{hf}(\mathbf{x},\mathbf{k})}{\partial x_i^2} - \alpha^2 \phi_{hf}(\mathbf{x},\mathbf{k}) \approx \iota J_0 k_1 \exp(\iota k_i x_i) \tag{29}$$

where J_0 is the constant mean head gradient. In an unbounded domain, the corresponding solution is,

$$\phi_{hf}(\mathbf{x},\mathbf{k}) = -\frac{\iota J_0 k_1}{k^2 + \alpha^2} \exp(\iota k_i x_i) \tag{30}$$

In turn, the head fluctuation is represented via equation (15) as

$$h^{(1)}(\mathbf{x}) = \int \phi_{hf}(\mathbf{x},\mathbf{k}) dZ_f(\mathbf{k}) = -\int \frac{\iota J_0 k_1}{k^2 + \alpha^2} \exp(\iota k_i x_i) dZ_f(\mathbf{k}) \tag{31}$$

This form of stationary head fluctuation expression is the same as that derived *Zhu* [1998, Eqs. 10 and 13] with the classical (stationary) spectral method, the essence of which is to apply the standard Fourier-Stieltjes representation (8)-(9) to (7) under the condition of $J_{x1}(x_1) \approx J_0$. It is seen from (31) and (25) that the nonstationary representation of the head fluctuation has an additional dependency on the mean relative head $h_r^0(x_1)$. With the stationary representation (31), the head covariance is obtained with (31) as,

$$C_{hh}(\mathbf{x}, \mathbf{x}') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_0^2 \frac{k_1^2}{(k^2 + \alpha^2)^2} S_{ff}(\mathbf{k}) \cos(k_1 \xi_1 + k_2 \xi_2) dk_1 dk_2 \tag{32}$$

where $\boldsymbol{\xi} = \mathbf{x'} - \mathbf{x}$ is a displacement vector with components ξ_1 and ξ_2. The specific discharge fluctuation is obtained by substituting (8) and (31) into (18),

$$q_i'(\mathbf{x}) = -K_0\{\int \exp(\iota k_i x_i)dZ_f(\mathbf{k})[-J_0\delta_{i1}] + \frac{\partial}{\partial x_i}[\int \frac{-\iota J_0 k_1}{(k^2+\alpha^2)}\exp(\iota k_i x_i)dZ_f(\mathbf{k})]\}$$ (33)

$$= K_0\{\int J_0[\delta_{i1} - \frac{k_1 k_i}{(k^2+\alpha^2)}]\exp(\iota k_i x_i)dZ_f(\mathbf{k})\}$$

where δ_{ij} is the Kronecker delta. With (33), we obtain the following covariance functions of specific discharge, after some manipulations,

$$\frac{C_{q_i q_j}(\mathbf{x},\mathbf{x'})}{K_0^2 J_0^2} = \int [\delta_{i1} - \frac{k_1 k_i}{(k^2+\alpha^2)}][\delta_{j1} - \frac{k_1 k_j}{(k^2+\alpha^2)}]\}S_{ff}(\mathbf{k})\cos(k_1\xi_1 + k_2\xi_2)d\mathbf{k}$$ (34)

The stationary results in (34) are, again, the same as those by *Zhu* [1998].

On the basis of these stationary expressions, *Zhu* [1998] and *Zhu and Sykes* [2000] have obtained analytical solutions for the head and specific discharge covariances. As discussed earlier, these stationary expressions are valid only in the limit of $\alpha \to 0$. It will be shown in Section 5 that the nonstationary results derived in the next section reduce to the stationary results.

4. Analytical Solutions

In this section, we evaluate the (co)variances of head and specific discharge analytically or semi-analytically on the basis of the nonstationary expressions given in Section 3.1. For a two-dimensional steady state flow problem, we use the following spectral density function to characterize the log hydraulic conductivity field,

$$S_{ff}(\mathbf{k}) = \frac{16\sigma_f^2 \lambda^5 k}{\pi^2 (k_1^2 + k_2^2 + \lambda^2)^4} \tag{35}$$

where σ_f^2 is the variance of f, λ is a parameter associated with the correlation scale l_e of the hydraulic conductivity field, $\lambda = 16/(3\pi\ l_e)$, and $k = \sqrt{k_1^2 + k_2^2}$. It is modified from the forms of spectrum for two-dimensional flow proposed by *Mizell et al*. [1982]. The auto correlation function in real space is plotted in Figure 3 and is compared with that of exponential form.

4.1. Head Covariance

With (35), the head covariance is obtained from (26), after some manipulations, as

$$C_{hh}(\mathrm{x},\ \mathrm{x}') = \frac{16\sigma_f^2 \lambda^5}{\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{k}{(k_1^2 + k_2^2 + \lambda^2)^4} \frac{k_1^2}{(k^4 + 4\alpha^2 k_1^2)^2} \{4\alpha^4 k_1^2 h_r^0(x_1) h_r^0(x_1') \\ + k^4 J_{x1}(x_1) J_{x1}(x_1')\} \cos(k_1\xi_1 + k_2\xi_2)\, dk_1 dk_2 \tag{36}$$

The head variance is obtained from the corresponding covariance function in (36) at zero lag distance, i.e., $\xi_1 = \xi_2 = 0$, and by evaluating the resulting integral in a polar coordinate system,

$$\frac{\sigma_h^2}{\sigma_f^2 J_0^2} = \frac{16\lambda^5}{\pi J_0^2} \int_0^{\infty} \{[h_r^0(x_1)]^2 \frac{k^2}{2(k^2 + \lambda^2)^4} [1 - \frac{2k^3 + 3\mu^2 k}{2(k^2 + \mu^2)^{3/2}}] \\ + [J_{x1}(x_1)]^2 \frac{k^3}{(k^2 + \lambda^2)^4 (k^2 + \mu^2)^{3/2}}\} dk \tag{37}$$

where μ=2α, and J_0=$J_{x1}(0)$. With *Mathematica* [*Wolfram*, 1991], we obtain the following closed form expression for the head variance from (37),

$$\frac{\sigma_h^2}{\sigma_f^2 J_0^2} = \frac{16\lambda^5}{\pi J_0^2}\{[h_r^0(x_1)]^2[\frac{\pi}{64\lambda^5} - \frac{1}{384\lambda^4(\lambda^2-\mu^2)^{9/2}}[(12\lambda^6\mu - 110\lambda^4\mu^3 - 136\lambda^2\mu^5$$
$$+24\mu^7)(\lambda^2-\mu^2)^{1/2} + (6\lambda^8 - 27\lambda^6\mu^2 + 126\lambda^4\mu^4)(\pi - \text{atan}(\frac{\mu}{(\lambda^2-\mu^2)^{1/2}})]$$
$$+[J_{x1}(x_1)]^2\frac{1}{96\lambda^4(\lambda^2-\mu^2)^{9/2}}[(-162\lambda^4\mu - 56\lambda^2\mu^3 + 8\mu^5)(\lambda^2-\mu^2)^{1/2}$$
$$+15(\lambda^6 + 6\lambda^4\mu^2)(\pi - \text{atan}(\frac{\mu}{(\lambda^2-\mu^2)^{1/2}})]\} \tag{38}$$

4.2. Covariance of Specific Discharge

After substituting (35) into (28), we get the following covariance functions of specific discharge, after some manipulations,

$$\frac{C_{q_1q_1}(\mathbf{x},\mathbf{x}')}{K_0^2\sigma_f^2 J_0^2} = \frac{16\lambda^5}{\pi^2 J_0^2}\int \frac{k}{(k^2+\lambda^2)^4}\frac{k_1^2}{(k^4+4\alpha^2k_1^2)^2}\{\alpha^4 h_r^0(x_1)h_r^0(x_1')(2k_1^2-k^2)^2$$
$$+J_{x1}(x_1)J_{x1}(x_1')(\frac{k^4+4\alpha^2k_1^2}{k_1} - k_1k^2 - 2\alpha^2k_1)^2\}\cos(k_1\xi_1)\cos(k_2\xi_2)d\mathbf{k} \tag{39}$$

$$\frac{C_{q_1q_2}(\mathbf{x},\mathbf{x}')}{K_0^2\sigma_f^2 J_0^2} = \frac{16\lambda^5}{\pi^2 J_0^2}\int \frac{k}{(k^2+\lambda^2)^4}\frac{k_1^2}{(k^4+4\alpha^2k_1^2)^2}\{\alpha^4 h_r^0(x_1)h_r^0(x_1')$$
$$[2k_1k_1 - k^2][2k_1k_2 - k^2] + J_{x1}(x_1)J_{x1}(x_1')[\frac{k^4+4\alpha^2k_1^2}{k_1}$$
$$-k_1k^2 - 2\alpha^2k_1][-k_2k^2 - 2\alpha^2k_1]\}\sin(k_1\xi_1)\sin(k_2\xi_2)d\mathbf{k} \tag{40}$$

$$\frac{C_{q_2q_2}(\mathbf{x},\mathbf{x}')}{K_0^2\sigma_f^2 J_0^2} = \frac{16\lambda^5}{\pi^2 J_0^2}\int \frac{k}{(k^2+\lambda^2)^4}\frac{k_1^2}{(k^4+4\alpha^2k_1^2)^2}\{$$
$$\alpha^4 h_r^0(x_1)h_r^0(x_1')[2k_1k_2 - k^2]^2$$
$$+J_{x1}(x_1)J_{x1}(x_1')[-k_2k^2 - 2\alpha^2k_1]^2\}\cos(k_1\xi_1)\cos(k_2\xi_2)d\mathbf{k} \tag{41}$$

The variance of the specific discharge in the longitudinal direction is obtained by letting $\mathbf{x}' = \mathbf{x}$ or $\xi_1 = \xi_2 = 0$ in (39) and evaluating the resulting integrals in a polar coordinate system:

$$\frac{\sigma_{q_1}^2}{K_0^2\sigma_f^2 J_0^2} = \frac{16\lambda^5}{\pi J_0^2}\int \frac{k^2}{(k^2+\lambda^2)^4}\{[h_r^0(x_1)]^2 \frac{1}{16\mu^2}(-16k^4 - 4\mu^2 k^2 + \frac{16k^7 + 28\mu^2 k^5 + 12\mu^4 k^3 + \mu^2 k}{(k^2+\mu^2)^{3/2}}) + [J_{x1}(x_1)]^2 \frac{1}{4\mu^4} [8\mu^4 - 8\mu^2(2k^2+\mu^2)(1 - \frac{k}{(k^2+\mu^2)^{1/2}}) + + 2(4k^4 + 4\mu^2 k^2 + \mu^4)(1 - \frac{2k^3 + 3\mu^2 k}{2(k^2+\mu^2)^{3/2}})]\}d\mathbf{k} \tag{42}$$

With Equation (42) being evaluated with *Mathematica* [*Wolfram*, 1991], we obtain the following closed-form expression for the normalized specific discharge variance in the longitudinal direction:

$$\begin{aligned}\frac{\sigma_{q_1}^2}{K_0^2\sigma_f^2 J_0^2} = \frac{16\lambda^5}{\pi J_0^2}\{&[h_r^0(x_1)]^2 \frac{1}{\mu^2}[\frac{-(20\lambda^2+\mu^2)\pi}{32\lambda^3} + \frac{1}{384\lambda^4(\lambda^2-\mu^2)^{9/2}}[(480\lambda^{10}\mu \\ &-1816\lambda^8\mu^3 + 2504\lambda^6\mu^5 - 1522\lambda^4\mu^7 + 136\lambda^2\mu^9 + 8\mu^{11})(\lambda^2-\mu^2)^{1/2} \\ &+(240\lambda^{12} - 1068\lambda^{10}\mu^2 + 1836\lambda^8\mu^4 - 1473\lambda^6\mu^6 + 570\lambda^4\mu^8)(\pi - \text{atan}(\frac{\mu}{(\lambda^2-\mu^2)^{1/2}})] + \\ &[J_{x1}(x_1)]^2 \frac{1}{4\mu^4}[\frac{(20\lambda^4 - 4\lambda^2\mu^2 + \mu^4)\pi}{16\lambda^5} + \frac{1}{96\lambda^4(\lambda^2-\mu^2)^{9/2}}[(-240\lambda^{10}\mu \\ &+968\lambda^8\mu^3 - 1476\lambda^6\mu^5 + 1038\lambda^4\mu^7 - 120\lambda^2\mu^9 + 40\mu^{11})(\lambda^2-\mu^2)^{1/2} \\ &+(-120\lambda^{12} + 564\lambda^{10}\mu^2 - 1050\lambda^8\mu^4 + 963\lambda^6\mu^6 - 462\lambda^4\mu^8)(\pi - \text{atan}(\frac{\mu}{(\lambda^2-\mu^2)^{1/2}})]\}\end{aligned} \tag{43}$$

Similarly, the variance in the transverse direction is derived as

$$\frac{\sigma_{q_2}^2}{K_0^2\sigma_f^2 J_0^2}=\frac{16\lambda^5}{\pi J_0^2}\int \frac{k^2}{(k^2+\lambda^2)^4}\{[h_r^0(x_1)]^2\frac{1}{4\mu^2}[16k^4+4\mu^2k^2 + \frac{-16k^7-28\mu^2k^5-12\mu^4k^3+\mu^6k}{(k^2+\mu^2)^{3/2}})]+ [J_{x1}(x_1)]^2\frac{1}{4\mu^4}[-8k^4+6\mu^4+\frac{8k^7+12\mu^2k^5+2\mu^4k^3-3\mu^6k}{(k^2+\mu^2)^{3/2}})]\}d\mathbf{k} \tag{44}$$

With *Mathematica* [*Wolfram*, 1991], from (44) we obtain the following closed-form expression for the normalized specific discharge variance in the transverse direction:

$$\begin{aligned}\frac{\sigma_{q_2}^2}{K_0^2\sigma_f^2 J_0^2}= &\frac{16\lambda^5}{\pi J_0^2}\{[h_r^0(x_1)]^2\frac{1}{4\mu^2}[\frac{(20\lambda^2+\mu^2)\pi}{8\lambda^3}+\frac{1}{96\lambda^4(\lambda^2-\mu^2)^{9/2}}[(-480\lambda^{10}\mu \\ &+1816\lambda^8\mu^3-2504\lambda^6\mu^5+1198\lambda^4\mu^7-248\lambda^2\mu^9+8\mu^{11})(\lambda^2-\mu^2)^{1/2} \\ &+(-240\lambda^{12}+1068\lambda^{10}\mu^2-1836\lambda^8\mu^4+1503\lambda^6\mu^6-390\lambda^4\mu^8)(\pi-\mathrm{atan}(\frac{\mu}{(\lambda^2-\mu^2)^{1/2}})] \\ &+[J_{x1}(x_1)]^2\frac{1}{4\mu^4}[\frac{(-20\lambda^4+\mu^4)\pi}{16\lambda^5}+\frac{1}{96\lambda^4(\lambda^2-\mu^2)^{9/2}}[(240\lambda^{10}\mu \\ &-920\lambda^8\mu^3+1292\lambda^6\mu^5-578\lambda^4\mu^7+200\lambda^2\mu^9-24\mu^{11})(\lambda^2-\mu^2)^{1/2} \\ &+(120\lambda^{12}-540\lambda^{10}\mu^2+942\lambda^8\mu^4-789\lambda^6\mu^6+162\lambda^4\mu^8)(\pi-\mathrm{atan}(\frac{\mu}{(\lambda^2-\mu^2)^{1/2}})]\}\end{aligned} \tag{45}$$

The covariances of head and specific discharge are computed from (36) and (39)-(41) by numerical integrations. The accuracy of the integration procedure is verified by comparing the variances obtained numerically with the closed-form solutions given in (38), (43) and (45).

5. Results and Discussion

In this section, we discuss some results pertinent to the variances and covariances of hydraulic head and specific discharge and their dependency on the leakage factor. In particular, we show how the results from the nonstationary approach compare with those from the stationary one. The flow domain and the mean (relative) head have already been discussed in Section 3 (see Figure 2). For the results discussed in the following, the correlation length l_e of (35) is set to *1.0* m and σ_f^2 is taken to be *1.0*. Following *Zhu* [1998], we define a dimensionless leakage factor $\gamma = \alpha\, l_e$. In the following examples, the values of γ range from 0 to 0.04, which are reasonable on the basis of our previous discussion and are smaller than the value of $\gamma = 1$ that *Zhu* [1998] used his stationary model.

Figures 4 and 5 show the head and specific discharge variances as functions of x_1 and as functions of the dimensionless leakage factor, respectively. As the mean flow is unidirectional along the x_1 direction and the effect of finite boundaries on the second moments is neglected, the variances do not vary in space along the transverse (x_2) direction. It is seen that for a dimensionless specific leakage the head variance increases longitudinally (Figure 4a). For $\gamma = 0.01$, the head variance is almost constant in space. The dependency of the head variance on the dimensionless leakage factor γ is a strong function of the location in the longitudinal direction. At $x_1=0$, the head variance initially decreases with γ but increases for $\gamma > 0.025$; at $x_1=5$ and 10, the head variance decreases only for very small γ (say, <0.004) and then increases with γ. This observation differs from that by *Zhu and Sykes* [2000] with a stationary approach: The head variance decreases monotonically with the increase of γ (see the dash-dot-dot curve in Figure 5a). *Zhu and Sykes* [p.205, 2000] attributed "this significant reduction in the head variance with increasing [leakage factor]" to "[for a large leakage factor] the large head above the leaky layer nullifies the head variation in the main aquifer". This apparent contradiction may be explained with a comparison based on (26) and (32). The nonstationary expression in (26) consists of two terms (one involving with J_{x1} and the other with h_r) while the stationary expression (36) has only one term involving with J_0. The contributions of the two terms in (26) are plotted in Figure 5b for $x_1=5$. It is seen that the J_{x1} related term decreases slowly while the h_r related term is zero for very small γ values and increases rapidly with γ for large γ values. Hence, the (total) head variance increases with γ except for very small γ values. This comparison suggests that the stationary approximation is valid for and only for very small leakage factor values.

The variances of the longitudinal and transverse specific discharge components vary spatially along the x_1 direction (Figures 4b, c). At $x_1=0$, the specific discharge variances decrease with the dimensionless leakage factor; at $x_1=10$, the variances increase with γ (Figures 5c,d). Only near $x_1=5.0$ (the center of the domain), the specific discharge variances are almost identical for the stationary and nonstationary approaches. This observation is different from that by *Zhu* [1998] with the stationary spectral method: The longitudinal and transverse specific discharge variances are monotonic (increasing and decreasing, respectively) functions of dimensionless leakage factor, independent of location in the domain. As γ decreases, the longitudinal and transverse specific discharge variances normalized with respect to $\sigma_f^2 J_0^2$ approach *0.375* and *0.125*, respectively. It is seen that in the limit of $\gamma \rightarrow 0$, our nonstationary results reduce to the well know results for two-dimensional, uniform mean flow [*Dagan*, 1989; *Rubin*, 1990; Zhang, 2002].

Figure 6 depicts the head covariance $C_{hh}(\mathbf{x}, \mathbf{x}')$ as a function of $\mathbf{x}(x_1, x_2)$ for dimensionless leakage factor $\gamma = 0$ and $\gamma = 0.04$, respectively. In Figure 6 (and also in Figures 7 and 8 to be discussed next), the reference point $\mathbf{x}'$ is selected to be the center of the domain. It is seen that the head covariance is asymmetric for $\gamma = 0.04$ in the longitudinal direction (and thus nonstationary) while it is symmetric for $\gamma = 0$. The covariances of longitudinal and transverse specific discharge are shown in Figures 7 and 8, respectively. Also, the nonstationary approach yields asymmetric specific discharge covariances for $\gamma = 0.04$ while the stationary approach [*Zhu*, 1998] produces symmetric (and stationary) ones.

6. Conclusions

In this study, we investigate flow in a heterogeneous, semiconfined (leaky) aquifer with a nonstationary spectral perturbation method. We evaluate the resulting moments of head and specific discharge analytically or semi-analytically and compare them with those derived with a stationary spectral method. The main findings of this study are summarized as follows:

1). Due to the spatial variation of the mean head gradient in a leaky aquifer, the flow field is nonstationary. This flow nonstationarity calls for a nonstationary stochastic approach as stationary approaches fail in such a situation. This study reveals that the nonstationary spectral method developed *by Li and McLaughlin* [1991, 1995] is applicable to such a flow in the case of stationary log transformed hydraulic conductivity.

2). The nonstationary formulations for flow in the semiconfined aquifer have terms associated with both the mean relative head and the mean head gradient. The contribution for the term associated with the mean relative head is significant for large values of the leakage factor and is exactly what the stationary approach by *Zhu* [1998] and *Zhu and Sykes* [2000] fails to capture. As expected, the effects of leakage on flow increase as the leakage factor increases.

3). The nonstationary and stationary approaches yield very different results for large values of the dimensionless leakage factor γ while they are identical in the limit of very small γ values. Unlike predicted by the stationary approach, the nonstationary head variance does not monotonically decrease with the increase of γ. The head variance decreases initially for very small γ but eventually increases with γ for larger γ values. Specific discharge variances, both the longitudinal and transverse components, decrease in the upstream of the flow domain but increase in the downstream domain as the dimensionless leakage factor γ increases. This observation is different from that by *Zhu* [1998] with the stationary spectral method: The specific discharge variances are monotonic functions of the dimensionless leakage factor, independent of location in the domain.

4). The covariance functions of head and specific discharge are normally asymmetric (and thus anisotropic as well as nonstationary) in the presence of leakage.

5). Both the non-uniform mean gradient and the presence of finite flow boundaries contribute to the nonstationary (location-dependent) behaviors of the head and specific discharge statistical moments. In this study, we have neglected the effects of finite boundaries on the head and specific discharge (co)variances for the purpose of deriving analytical or semi-analytical solutions. To fully account for the latter effects, a numerically based moment equation approach is appropriate as discussed by *Zhang* [2002] and is out of the scope of this study.

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Captions

Figure 1. A schematic semiconfined leaky aquifer.

Figure 2. (a) The mean relative head, and (b) mean head gradient along the longitudinal direction for selected values of leakage factor.

Figure 3. Autocorrelation function of the spectrum function of log transformed hydraulic conductivity.

Figure 4. Variances along the mean flow direction for dimensionless leakage factor $\gamma = 0, 0.01, 0.02$ and 0.04: (a) head variance, (b) longitudinal specific discharge variance, and (c) transverse specific discharge variance.

Figure 5. Variances as functions of dimensionless leakage factor at $x_1 = 0, 5$, and $10\ m$: (a) head variance, (b) head variance at $x_1 = 5\ m$ with contributions from h_r and J_x related terms, (c) longitudinal specific discharge variance, and (d) transverse specific discharge variance.

Figure 6. Head covariance contours for (a) $\gamma = 0.0$, and (b) $\gamma = 0.04$.

Figure 7. Longitudinal specific discharge covariance contours for (a) $\gamma = 0.0$, and (b) $\gamma = 0.04$.

Figure 8. Transverse specific discharge covariance contours for (a) $\gamma = 0.0$, and (b) $\gamma = 0.04$.

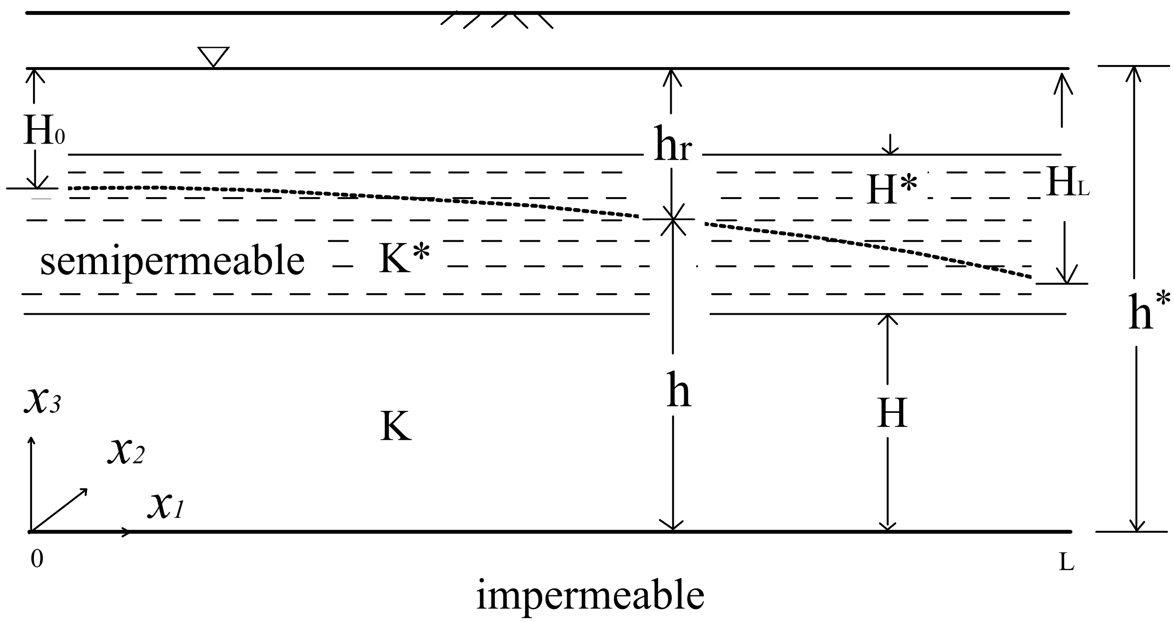
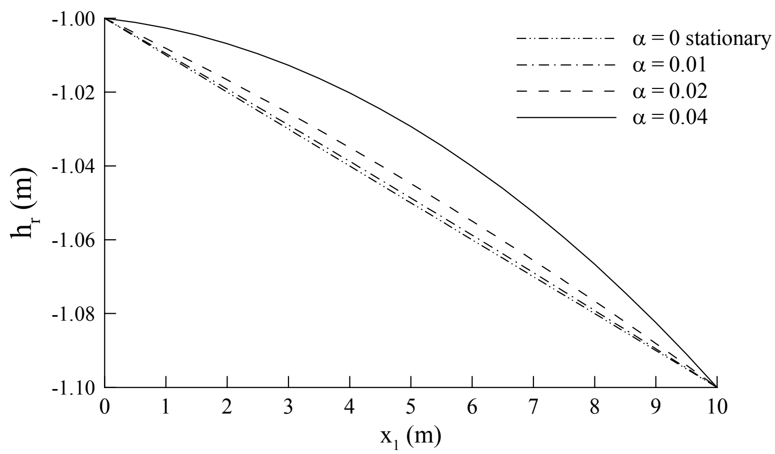
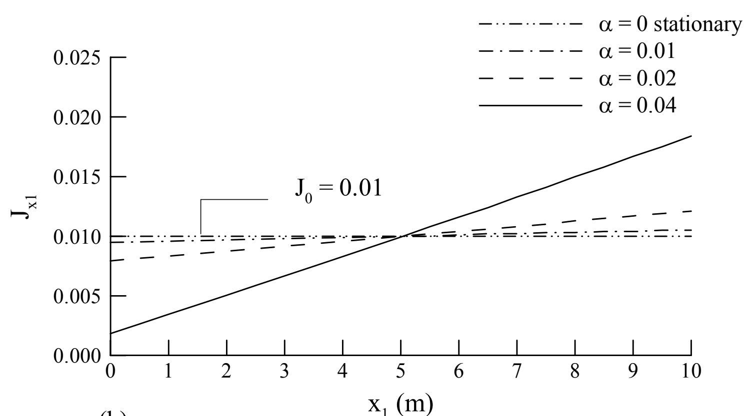


Figure 1


-1.00
-1.02
-1.04
-1.06
-1.08
-1.10
0 1 2 3 4 5 6 7 8 9 10
h_r (m)
x_1 (m)
α = 0 stationary
α = 0.01
α = 0.02
α = 0.04

(a)


0.025
0.020
0.015
0.010
0.005
0.000
0 1 2 3 4 5 6 7 8 9 10
J_{x1}
x_1 (m)
J_0 = 0.01
α = 0 stationary
α = 0.01
α = 0.02
α = 0.04

(b)

Figure 2

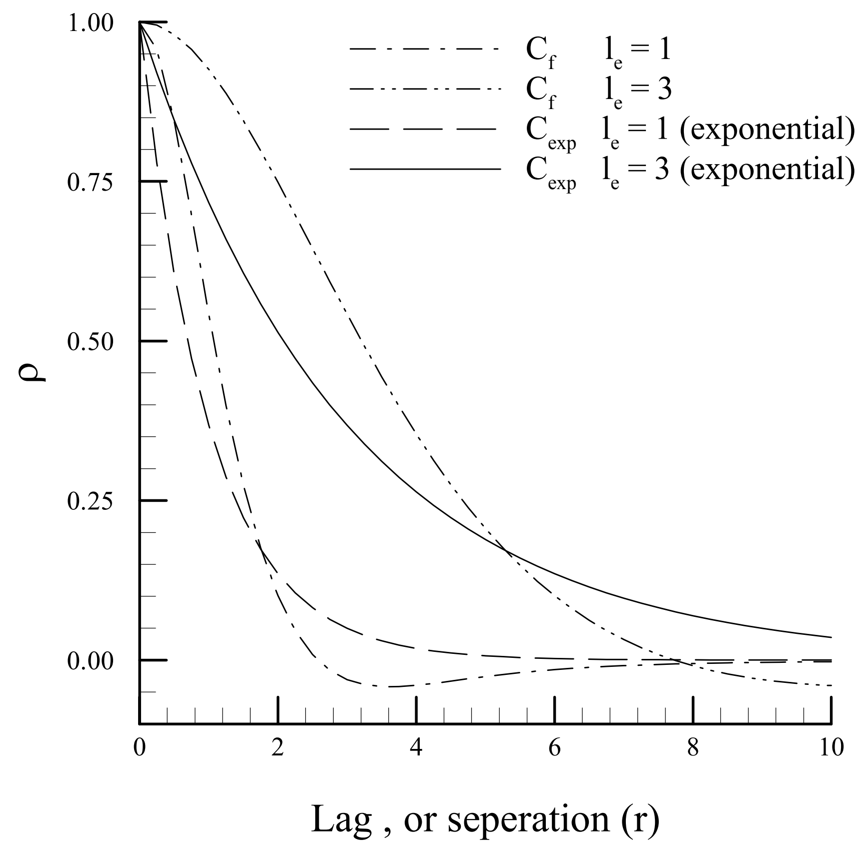
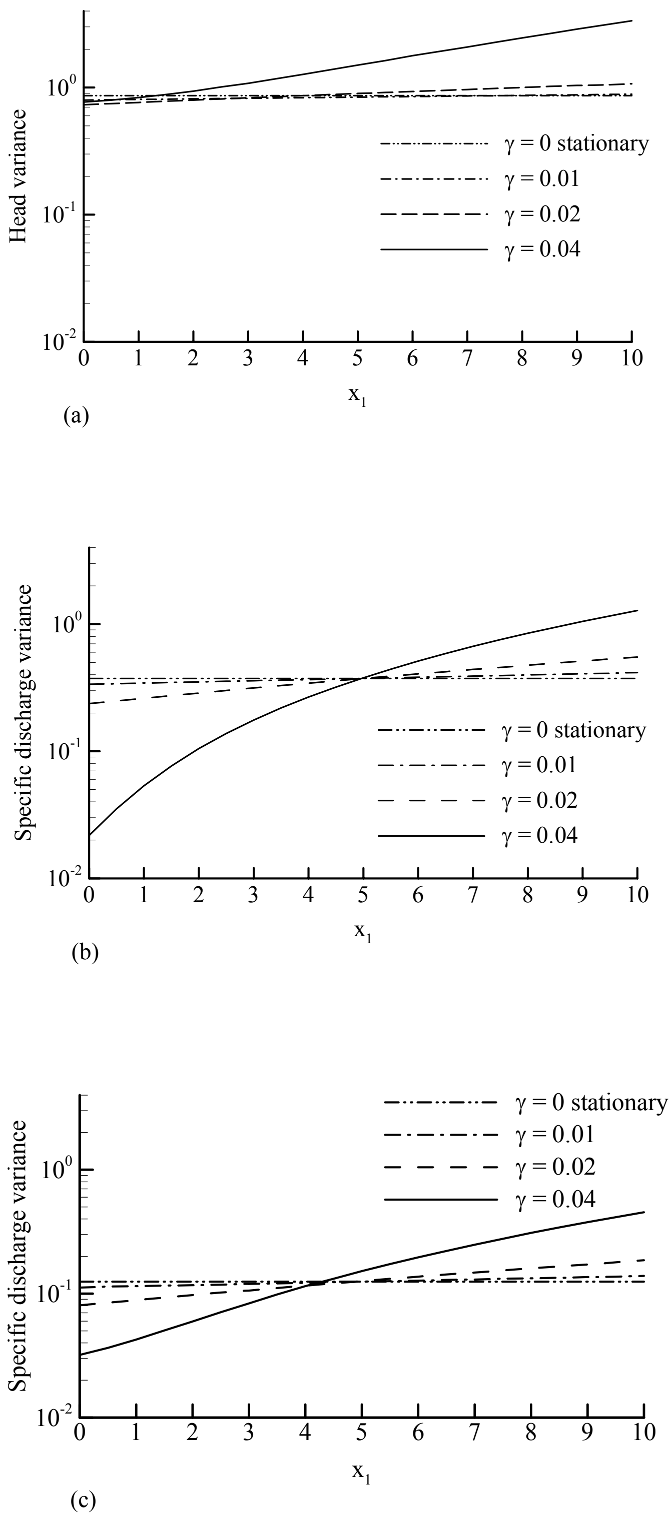
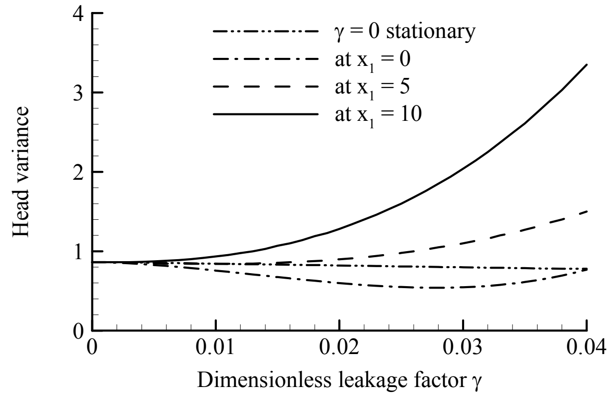

1.00
0.75
0.50
0.25
0.00
ρ
0
2
4
6
8
10
C_f $l_e = 1$
C_f $l_e = 3$
C_{exp} $l_e = 1$ (exponential)
C_{exp} $l_e = 3$ (exponential)
Lag , or seperation (r)

Figure 3

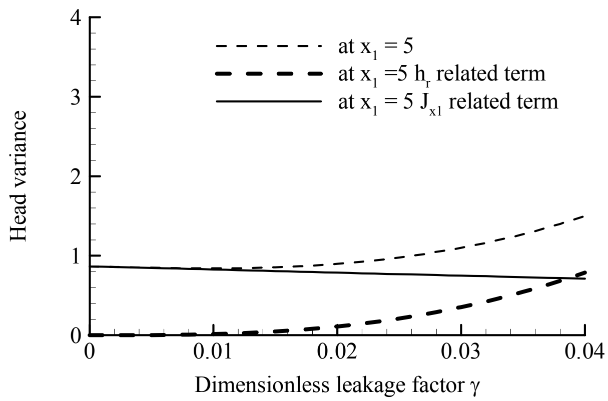


Head variance
10^{0}
10^{-1}
10^{-2}
$\gamma = 0$ stationary
$\gamma = 0.01$
$\gamma = 0.02$
$\gamma = 0.04$
0 1 2 3 4 5 6 7 8 9 10
x_1
(a)
Specific discharge variance
10^{0}
10^{-1}
10^{-2}
$\gamma = 0$ stationary
$\gamma = 0.01$
$\gamma = 0.02$
$\gamma = 0.04$
0 1 2 3 4 5 6 7 8 9 10
x_1
(b)
Specific discharge variance
$\gamma = 0$ stationary
$\gamma = 0.01$
$\gamma = 0.02$
$\gamma = 0.04$
10^{0}
10^{-1}
10^{-2}
0 1 2 3 4 5 6 7 8 9 10
x_1
(c)

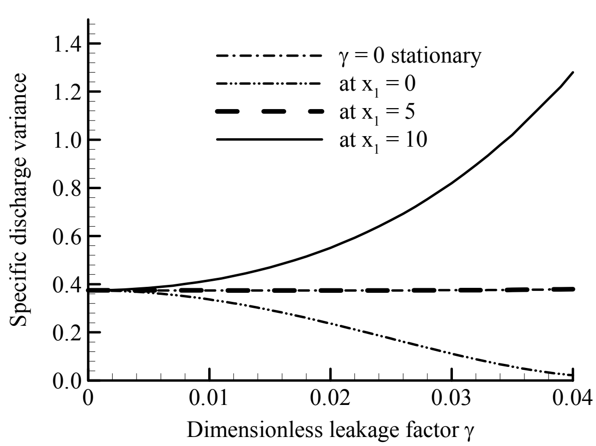
Figure 4


γ = 0 stationary
at x1 = 0
at x1 = 5
at x1 = 10
Head variance
Dimensionless leakage factor γ
0
1
2
3
4
0.01
0.02
0.03
0.04

(a)

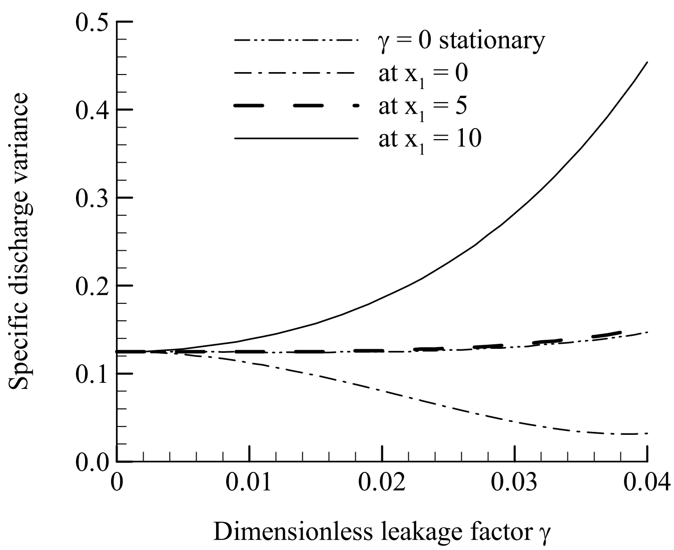

at x1 = 5
at x1 =5 hr related term
at x1 = 5 Jx1 related term
Head variance
Dimensionless leakage factor γ
0
1
2
3
4
0.01
0.02
0.03
0.04

(b)


γ = 0 stationary
at x1 = 0
at x1 = 5
at x1 = 10
Specific discharge variance
Dimensionless leakage factor γ
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
0
0.01
0.02
0.03
0.04

(c)

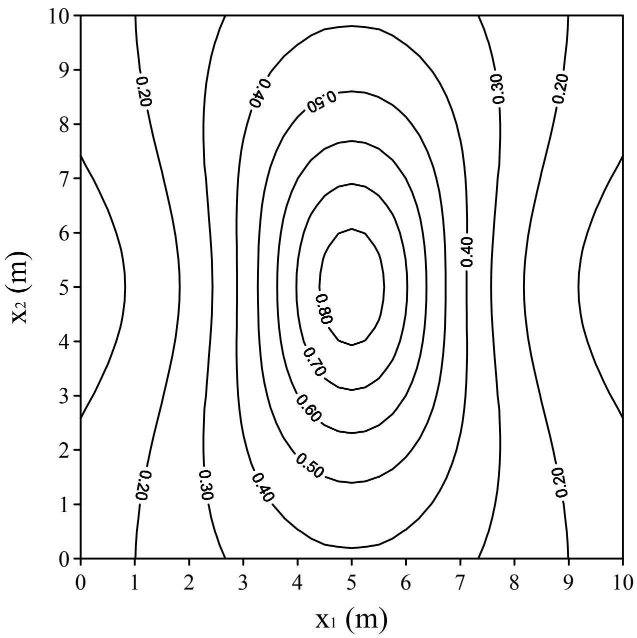
Figure 5



γ = 0 stationary
at x1 = 0
at x1 = 5
at x1 = 10
Specific discharge variance
Dimensionless leakage factor γ
0.5
0.4
0.3
0.2
0.1
0.0
0
0.01
0.02
0.03
0.04

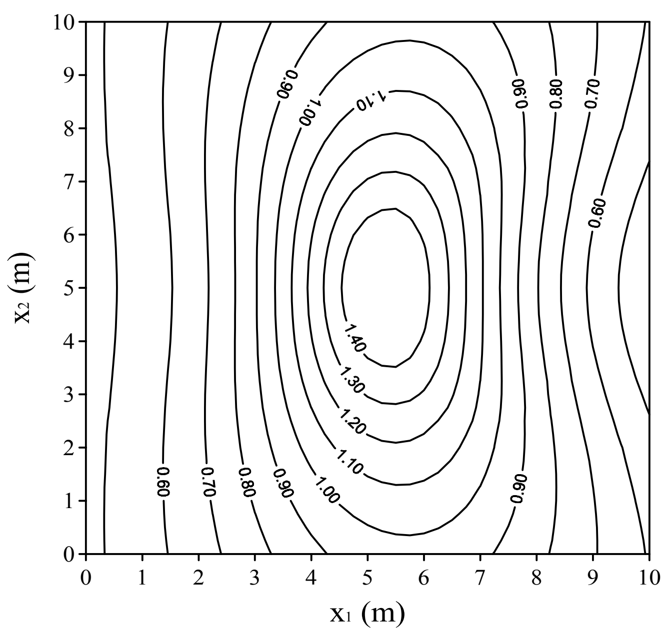
(d)

Figure 5 (d)



x2 (m)
x1 (m)
0.20
0.30
0.40
0.50
0.60
0.70
0.80

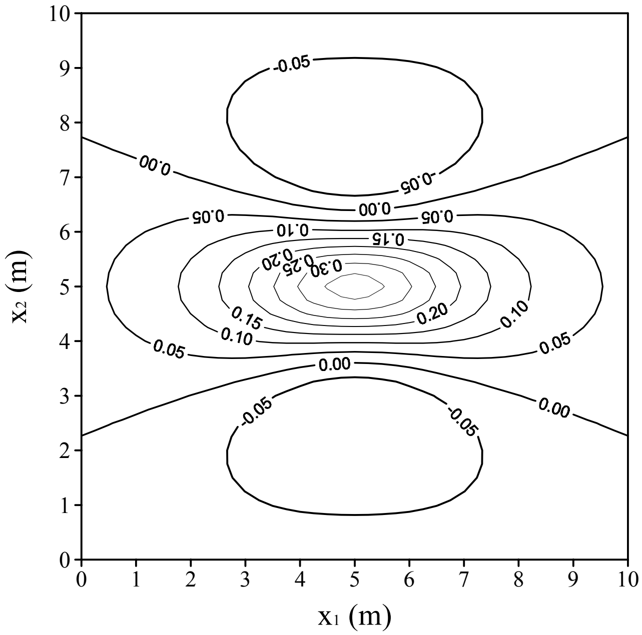
(a)



x2 (m)
x1 (m)
0.60
0.70
0.80
0.90
1.00
1.10
1.20
1.30
1.40

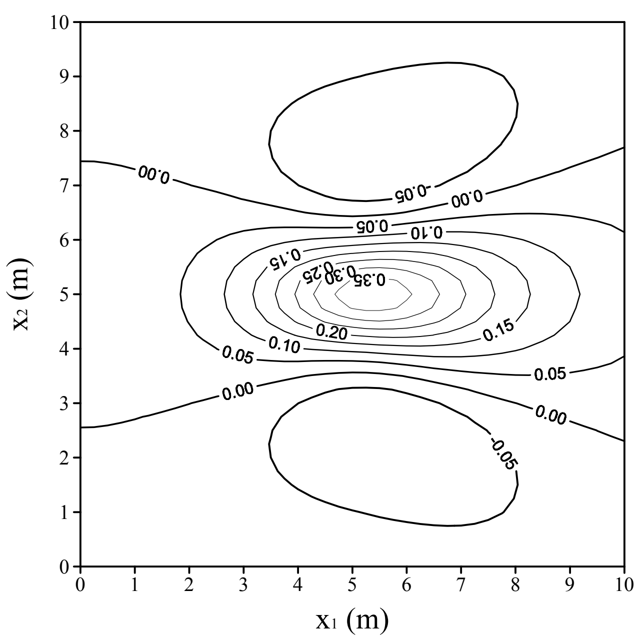
(b)

Figure 6



x₂ (m)
x₁ (m)
-0.05
0.00
0.05
0.10
0.15
0.20
0.25
0.30

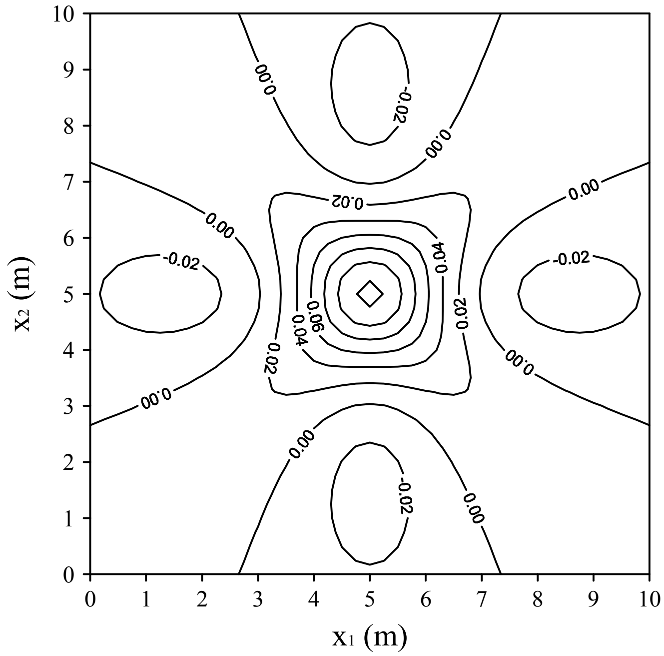
(a)



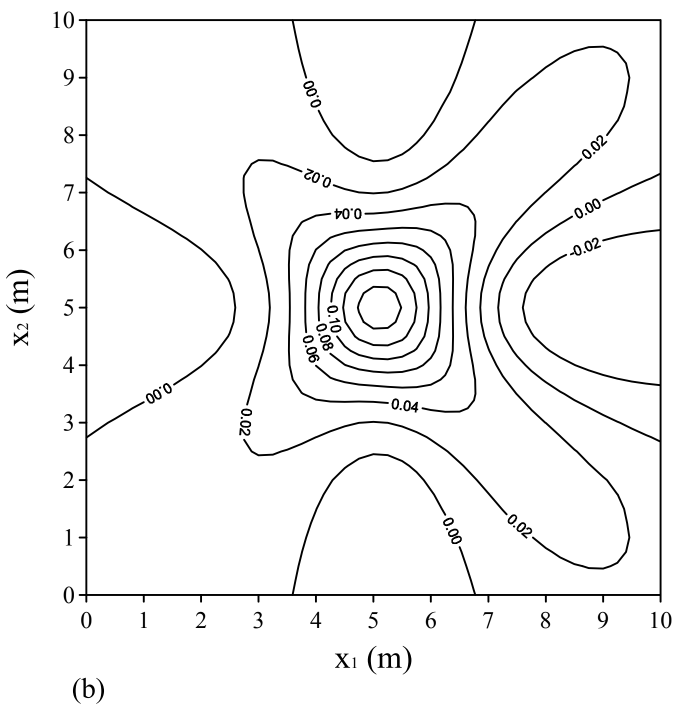
x₂ (m)
x₁ (m)
-0.05
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35

(b)

Figure 7



(a)



(b)

Figure 8